Amalgamated Worksheet # 4

Various Artists

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1 Peyam Tabrizian

Problem 1:

Suppose V is a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Show that if every eigenvalue of T is real, then T is self-adjoint.

Solution: By the complex spectral theorem, there exists an orthonormal basis $\mathcal{B} = (v_1, \cdots, v_n)$ of V consisting of eigenvectors of T.

Let v be an arbitrary vector in V.

By the above, we know that $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_1, \cdots, a_n$, where $T(v_i) = \lambda_i v_i$ for *real* eigenvalues λ_i $(i = 1, \cdots, n)$

In particular, since T^* is **normal**[1]

But then:

$$\begin{aligned}
T^*(v) &= T^*(a_1 v_1 + \cdots + a_n v_n) \\
&= a_1 T^*(v_1) + \cdots + a_n T^*(v_n) \\
&= a_1 \overline{\lambda_1} v_1 + \cdots + a_n \overline{\lambda_n} v_n \\
&= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \quad \text{because each } \lambda_i \text{ is real} \\
&= a_1 T(v_1) + \cdots + a_n T(v_n) \\
&= T(a_1 v_1 + \cdots + a_n v_n) \\
&= T(v)
\end{aligned}$$

[1] **WARNING:** It is not true **IN GENERAL**, that if $T(v_i) = \lambda_i v_i$ we have $T^*(v_i) = \overline{\lambda_i} v_i$, but it **IS** true for normal and self-adjoint operators!

So $T^*(v) = T(v)$ for all v, so $\boxed{T = T^*}$ $\square$

Problem 2:

Suppose A is a symmetric matrix over $\mathbb{R}$. Show that if $|\lambda| = 1$ for every eigenvalue λ of A, then $A^2 = I$.

First of all, $A^* = A^T$ (because A has only real entries) $= A$ (because A is symmetric), so $A^* = A$, so A is self-adjoint.

Hence, by the *real* spectral theorem (which holds for matrices as well), there exists an orthonormal basis $(v_1, \cdots, v_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of A. In particular, $Av_i = \lambda_i v_i$ for real eigenvalues λ_i $(i = 1, \cdots, n)$

Now let x, y be arbitrary vectors in $\mathbb{R}^n$.

Then $x = a_1v_1 + \cdots + a_nv_n$ and $y = b_1v_1 + \cdots + b_nv_n$ for scalars $a_1, \cdots, a_n, b_1, \cdots, b_n$.

<u>Goal:</u> We want to show $< A^2x, y >=< x, y >$

Now consider:

$$
\begin{aligned}
< A^2x, y > &= < A(Ax), y > \\
&= < Ax, A^*y > \\
&= < Ax, Ay > \quad \text{because } A^* = A \\
&= < A(a_1v_1 + \cdots + a_nv_n), A(b_1v_1 + \cdots + b_nv_n) > \\
&= < a_1A(v_1) + \cdots + a_nA(v_n), b_1A(v_1) + \cdots + b_nA(v_n) > \\
&= < a_1\lambda_1v_1 + \cdots + a_n\lambda_nv_n, b_1\lambda_1v_1 + \cdots + b_n\lambda_nv_n > \\
&= \begin{array}{ccc}
< a_1\lambda_1v_1, b_1\lambda_1v_1 > + & \cdots & + < a_1\lambda_1v_1, b_n\lambda_nv_n > \\
\vdots & < a_i\lambda_iv_i, b_j\lambda_jv_j > & \vdots \\
< a_n\lambda_nv_n, b_1\lambda_1v_1 > + & \cdots & + < a_n\lambda_nv_n, b_n\lambda_nv_n >
\end{array} \\
&= \sum_{i,j=1}^{n} < a_i\lambda_iv_i, b_j\lambda_jv_j >
\end{aligned}
$$

(In the middle step, we used distributivity/foil)

But because b_j and λ_j are real:

$$< a_i\lambda_i v_i, b_j\lambda_j v_j > = \ a_i\lambda_i\overline{b_j\lambda_j} < v_i, v_j > = \ a_i\lambda_i b_j\lambda_j < v_i, v_j >$$

Now if $i \neq j$, $< v_i, v_j >= 0$ (by orthogonality), hence $< a_i\lambda_i v_i, b_j\lambda_j v_j >= 0$

And if $i = j$, $< v_i, v_j >= 1$ (by orthonormality), so:

$$< a_i\lambda_i v_i, b_j\lambda_j v_j >= a_i\lambda_i b_i\lambda_i(1) = a_i b_i(\lambda_i)^2 = a_i b_i$$

(where we used $(\lambda_i)^2 = (|\lambda_i|)^2 = 1^2 = 1$, because λ_i is real)

Therefore, the above sum simplifies to:

$$< A^2x, y >= \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n$$

Similarly, we have:

$$\begin{aligned}
< x, y > &= < a_1 v_1 + \cdots + a_n v_n, b_1 v_1 + \cdots + b_n v_n > \\
&= \begin{matrix} < a_1 v_1, b_1 v_1 > + & \cdots & + < a_1 v_1, b_n v_n > \\ \vdots & < a_i v_i, b_j v_j > & \vdots \\ < a_n v_n, b_1 v_1 > + & \cdots & + < a_n v_n, b_n v_n > \end{matrix} \\
&= \sum_{i,j=1}^{n} < a_i v_i, b_j v_j > \\
&= \sum_{i,j=1}^{n} a_i b_j < v_i, v_j > \\
&= \sum_{i=1}^{n} a_i b_i \\
&= a_1 b_1 + \cdots + a_n b_n
\end{aligned}$$

Therefore:

$$< A^2x, y >= a_1 b_1 + \cdots + a_n b_n =< x, y >$$

Hence $< A^2x, y >=< x, y >$ for all x, y.

Hence $< A^2x - x, y >= 0$ for all y

Hence $A^2x - x = 0$ for all x, so $A^2x = x$ for all x

Hence $\boxed{A^2 = I}$ □

Problem 3:

(if time permits) If $U = \mathcal{P}_2(\mathbb{R})$, what is its complexification $U_{\mathbb{C}}$? Suppose $S \in \mathcal{L}(U)$ is defined by $S(p) = p'$, what is its complexification $S_{\mathbb{C}}$?

$$U_{\mathbb{C}} = U \times U = (\mathcal{P}_2(\mathbb{R})) \times (\mathcal{P}_2(\mathbb{R}))$$

Where we identify (p, q) with $p + iq$.

Moreover, addition in $U_{\mathbb{C}}$ is given by:

$$(p + iq) + (r + is) = (p + r) + i(q + s)$$

And scalar multiplication in $U_{\mathbb{C}}$ is given by:

$$(a + ib)(p + iq) = (ap - bq) + i(aq + bp)$$

(where $a, b \in \mathbb{R}$)

More precisely, a typical element of $U_{\mathbb{C}}$ is of the form:

$$a + bx + cx^2 + i(d + ex + fx^2) = (a + id) + (b + ie)x + (c + if)x^2 = c_1 + c_2x + c_3x^2$$

where $c_1 = a + id, c_2 = b + ie, c_3 = c + if$ are **complex** numbers

Conversely, if c_1, c_2, c_3 are arbitrary complex numbers, you can write $c_1 + c_2x + c_3x^2$ in the form $a + bx + cx^2 + i(d + ex + fx^2)$, where $a = Re(c_1)$, $b = Re(c_2)$, $c = Re(c_3)$, $d = Im(c_1)$, $e = Im(c_2)$, $f = Im(c_3)$.

It follows that $U_{\mathbb{C}}$ is in fact equal to the set of all polynomials in $\boxed{x}$ of degree 2 or less with coefficients in $\mathbb{C}$[2]

Finally:

[2]Note carefully that this is not equal to $\mathcal{P}_2(\mathbb{C})$, which is the set of all polynomials in $\boxed{z}$ of degree 2 or less with coefficients in $\mathbb{C}$. Here z is a *complex* variable whereas x is a *real* variable

$$\begin{aligned}
S_{\mathbb{C}}(p+iq) =& S_{\mathbb{C}}(a+bx+cx^2+i(d+ex+fx^2)) \\
=& S(a+bx+cx^2)+iS(d+ex+fx^2) \\
=& b+2cx+i(e+2fx) \\
=& (b+ie)+2(c+if)x \\
=& p'+iq' \\
=& (p+iq)'
\end{aligned}$$

That is, $S_{\mathbb{C}}(p)$ is still equal to p', except that now we're considering polynomials with coefficients in $\mathbb{C}$ instead of $\mathbb{R}$. □

Problem 4:

Briefly explain how to prove the real spectral theorem from the complex spectral theorem

Step 1: Complexify U and S to get $U_{\mathbb{C}}$ (which is a vector space over $\mathbb{C}$) and $S_{\mathbb{C}}$ (as in problem 3)

One can show that $S_{\mathbb{C}}$ is self-adjoint (and hence all its eigevalues are real).

Step 2: Apply the **complex** spectral theorem (which applies here) to $U_{\mathbb{C}}$ and $S_{\mathbb{C}}$ to obtain an orthonormal basis $(w_1, \cdots, w_n) = (u_1+iv_1, \cdots, u_n+iv_n)$ of $U_{\mathbb{C}}$ consisting of eigenvectors of $S_{\mathbb{C}}$

Step 3: One can show that in fact $u_1, \cdots, u_n, v_1, \cdots, v_n$ are eigenvectors of S, and in fact:

$$Span(u_1, \cdots, u_n, v_1, \cdots, v_n) = U$$

In particular, this implies that the span of all the eigenvectors of S is U $(\star)$

Now for each eigenvalue λ, find a basis for $Nul(S-\lambda I)$ and apply Gram-Schmidt to get an *orthonormal basis* for $Nul(S-\lambda I)$. Put all your basis vectors together to get a set $\mathcal{B}$

1) $\mathcal{B}$ is linearly independent because eigenvectors of S corresponding to distinct eigenvalues are linearly independent, and by construction

2) $\mathcal{B}$ spans U by $(\star)$

3) $\mathcal{B}$ is orthonormal because S is normal, and hence eigenvectors corresponding to distinct eigenvalues of S are orthonormal[3]

Hence $\mathcal{B}$ is an orthonormal basis of U consisting of eigenvectors of S $\square$

2 Daniel Sparks

1. Prove some of the properties of adjoints listed on p.119; S, T are operators on a finite dimensional complex vector space V.

   (a) $(S+T)^* = S^* + T^*$

   (b) $(aT)^* = \overline{a}T^*$

   (c) $(T^*)^* = T$

   Axler suggests thinking about $T \mapsto T^*$ as a function $* : \mathcal{L}(V) \to \mathcal{L}(V)$.

   (f) Look up the definition of a $\mathbf{C}^*$-algebra.

   (g) Show that $*$ is an isomorphism of $\mathbf{R}$ vector spaces.

   Solution: (a) $\langle (S+T)v, w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle = \langle v, S^*w\rangle + \langle v, T^*w\rangle = \langle v, (S^*+T^*)w\rangle$ for all $v, w \in V$. Hence $(S+T)^*(w) = (S^*+T^*)(w)$ for all $w \in V$.

   (b) $\langle (aT)v, w\rangle = \langle a(Tv), w\rangle = a\langle Tv, w\rangle = a\langle v, T^*w\rangle = \langle v, \overline{a}T^*w\rangle = \langle v, (\overline{a}T^*)w\rangle$ for all v, w. Hence $(aT)^*(w) = (\overline{a}T^*)(w)$ for all w.

   (c) $\langle T^*v, w\rangle = \overline{\langle w, T^*v\rangle} = \overline{\langle Tw, v\rangle} = \langle v, Tw\rangle$ for all v, w. Hence $(T^*)^*(w) = T(w)$ for all w.

   (f) A $\mathbf{C}^*$ algebra is something a lot like $\mathcal{L}(V)$ with its involution $T \mapsto T^*$.

   (g) Define $\Phi : \mathcal{L}(V) \to \mathcal{L}(V)$ by $\Phi(T) = T^*$. By (a), this map is additive. By (b), it is $\mathbf{R}$-linear: $\Phi(aT) = \overline{a}T^* = aT^* = a\Phi(T)$. By (c), $\Phi^2 = \mathrm{Id}_{\mathcal{L}(V)}$ so that Φ is its own inverse and, in particular, is invertible.

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[3]Proof: $< S(v_i), v_j > = < \lambda_i v_i, v_j > = \lambda_i < v_i, v_j >$, but also $< S(v_i), v_j > = < v_i, S^*(v_j) > = < v_i, \overline{\lambda_j} v_j > = \overline{\overline{\lambda_j}} < v_i, v_j > = \lambda_j < v_i, v_j >$. Hence $\lambda_i < v_i, v_j > = \lambda_j < v_i, v_j >$, hence $(\lambda_i - \lambda_j) < v_i, v_j > = 0$, and so $< v_i, v_j > = 0$ because $\lambda_i \neq \lambda_j$. See also the warning on page 1

2. Prove in detail DWD Lemma 7.1. Namely, if T is normal on a finite dimensional complex vector space V, then $\mathrm{Null}(T) = \mathrm{Null}(T^*)$.

Solution: Let V be finite dimensional over $\mathbf{C}$ and let $T \in \mathcal{L}(V)$ be normal. Suppose that $T(v) = 0$ for some nonzero v. Then $\langle T^*v, T^*v \rangle = \langle v, TT^*v \rangle = \langle v, T^*Tv \rangle = \langle v, T^*(0) \rangle = 0$, that is to say $\|T^*v\| = 0$, meaning $T^*v = 0$. This shows that $\mathrm{Null}\, T \subseteq \mathrm{Null}\, T^*$. Since this is valid for any normal T on V, in particular we can apply it to the adjoint operator T^*; we see then that $\mathrm{Null}\, T^* \subseteq \mathrm{Null}(T^*)^* = \mathrm{Null}\, T$ by 1.(c).

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 complex matrix. This determines a linear map $L_A : \mathbf{C}^2 \to \mathbf{C}^2$ by $v \mapsto Av$. Using only the identity

$$\langle L_A v, w \rangle = \langle v, L_A^* w \rangle$$

show that $L_A^* = L_{\overline{A}^t}$, where $\overline{A}^t = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$. "The adjoint of a matrix operator is the conjugate transpose." (For the ambitious, try an $n \times n$.)

Solution: Let L_A correspond to the $n \times n$ matrix (a_{ij}). Write (b_{ij}) for the matrix of the operator L_A^* in the standard basis. Then $(L_A^*)(e_j) = \sum_k b_{kj} e_k$, so

$$\begin{aligned}
\langle L_A^*(e_j), e_i \rangle &= \left\langle \sum_k b_{kj} e_k, e_i \right\rangle \\
&= \sum_k b_{kj} \langle e_k, e_i \rangle \\
&= b_{ij}
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\langle L_A^*(e_j), e_i \rangle &= \langle e_j, L_A(e_i) \rangle \\
&= \left\langle e_j, \sum_k a_{ki} e_k \right\rangle \\
&= \sum_k \overline{a_{ki}} \langle e_j, e_k \rangle \\
&= \overline{a_{ji}}
\end{aligned}$$

so $b_{ij} = \overline{a_{ji}}$ as desired.

4. Fill in the blanks. Let T be a normal operator on a finite dimensional $\mathbf{C}$-vector space V. Consider the list of distinct eigenvalues of T: $\lambda_1, \cdots, \lambda_m$. (This list is

nonempty becauase (a) every operator on a finite dimensional complex vector space has an eigenvalue.)

Let U_{λ_i} be the (b) generalized eigenspace corresponding to λ_i. By (c) the Jordan theorem, we have a decomposition $V = U_{\lambda_1} \oplus \cdots \oplus U_{\lambda_m}$.

Let $e_i = \dim U_{\lambda_i}$ be the (d) multiplicity of λ_i. Then we have bases $\beta_i' = \{u_{i,1}', \cdots, u_{i,e_i}'\}$ for each U_{λ_i}. We may use the (e) Gram-Schmidt process to obtain orthonormal bases $\beta_i = \{u_{i,1}, \cdots, u_{i,e_i}\}$ of each U_{λ_i}. We know that the concatenated list $\beta = (\beta_1, \cdots, \beta_m) = \{u_{1,1}, u_{1,2} \cdots, u_{1,e_1}, u_{2,1}, \cdots, u_{m,e_m}\}$ is a basis for V because (f) of a homework exercise on bases and direct sums. Notice that each of these basis vectors are normal (i.e. of norm 1) generalized eigenvectors, and that $u_{i,j} \perp u_{k,l}$ whenever $i = k$.

Now, since T is normal, because (g*) DWD Prop 7.2 we know that β is actually a basis of eigenvectors. Finally, because (h*) DWD Prop 7.4 we know that $u_{i,j} \perp u_{k,l}$ whenever $i \neq k$. Therefore β is an orthonormal eigenbasis.

* These results do not have names, but can be found in DWD.

5. Review/redo carefully the proofs of the results cited in (g) and (h) of the previous exercise.

   Solution: The proofs themselves are in DWD, the exercise is to do them yourself. Me writing them again here wont help with that.